MATH 20D: Differential Equations Spring 2023
Homework 3

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Make sure you show all your workings.
Points will be awarded for clear explanations, not just for arriving at the correct solution.
Remember to list the sources you used when completing the assignment.
Below *NSS* is used to reference the text *Fundamentals of Differential Equations (9th edition) by Nagle, Saff, Snider*

Question (1). *In parts (a)-(c) below, find constants r_1 and r_2 such that the ODE admits a general solution of the form*

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Then calculate the values of C_1 and C_2 so that $y(0) = 1$ and $y'(0) = 0$.

(a) $y'' - 3y' + 2y = 0$ ***(b)*** $y'' + 5y' + 6y = 0$ ***(c)*** $6y'' + y' - 2y = 0$

Question (2). *In parts (a)-(c) below, determine a single constant r such that the ODE admits a general solution of the form*

$$y(x) = C_1 e^{rt} + C_2 t e^{rt}.$$

Then calculate the values of C_1 and C_2 so that $y(0) = 1$ and $y'(0) = 1$.

(a) $4y'' - 4y' + y = 0$ ***(b)*** $4y'' + 20y' + 25y = 0$ ***(c)*** $y'' + 6y' + 9y = 0$

Question (3). *In parts (a)-(c) below, determine constants α and β so that the ODE admits a general solution of the form*

$$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$

Then calculate the values of C_1 and C_2 so that $y(0) = 1$ and $y'(0) = -1$.

(a) $y'' + 9y = 0$ ***(b)*** $y'' - 6y' + 20y = 0$ ***(c)*** $y'' - y' + 7y = 0$

Question (4). *In parts (a)-(c) below, solve the initial value problem expressing your solution in form*

$$y(x) = A e^{\alpha x} \sin(\beta x + \phi)$$

where $A > 0$ and $\phi \in [0, 2\pi)$ are constants.

(a) $y'' + 9y = 0$, $y(0) = 1$ *and* $y'(0) = -1$.
(b) $y'' - 6y' + 20y = 0$, $y(0) = -1$, $y'(0) = 1$.
(c) $y'' - y' + 7y = 0$, $y(0) = -1$, $y'(0) = -1$.

Question (5).
(a) Find a general solution to the ODE

$$y'' - y = 0. \tag{0.1}$$

Express your general solution in the form $y(x) = C_1e^{r_1 t} + C_2e^{r_2 t}$ *then calculate the values of* C_1 *and* C_2 *such that*

$$\textbf{\textit{(i)}}\ y(0) = 1, \quad y'(0) = 0, \qquad \textbf{\textit{(ii)}}\ y(0) = 0, \quad y'(0) = 1$$

(b) Let $\cosh(t)$ *denote the solution to the intial value problem (i) and let* $\sinh(t)$ *denote the solution to the intial value problem (ii). Using the explicit forms of the solutions you constructed in (a), show that*

$$\frac{d}{dt}(\cosh(t)) = \sinh(t) \qquad \textit{and} \qquad \frac{d}{dt}(\sinh(t)) = \cosh(t).$$

Then analyze the derivative $\frac{d}{dt}(\cosh^2(t) - \sinh^2(t))$ *to deduce that*

$$\cosh^2(t) - \sinh^2(t) = 1$$

(c) Let $\beta \neq 0$ *be constant. Show that* $\cosh(\beta t)$ *and* $\sinh(\beta t)$ *are linearly independent as functions on* $(-\infty, \infty)$.
Hint: The test for linear independence given in lecture 8 may be useful here.
(d) Let $\beta \neq 0$ *be constant. Explain why the ODE*

$$y'' - \beta^2 y = 0 \tag{0.2}$$

admits a general solution of the form $C_1 \cosh(\beta t) + C_2 \sinh(\beta t)$.
Hint: Combine the result of the previous part with the Theorem on slide 11 of lecture 7.
(e) Show that $\cosh(2t) = \cosh^2(t) + \sinh^2(t)$ *for all* $t \in \mathbb{R}$.
Hint: Solve the initial value problem

$$y'' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

using the $\beta = 2$ *case of the general solution you constructed in (d). Now verify that* $y(t) = \cosh^2(t) + \sinh^2(t)$ *also satisfies the IVP. Conclude by using the fact that solutions to IVPs are unique!*

Question (6). *(NSS 4.9.2) A 2kg mass is attached to a spring with stiffness* $k = 50N/m$. *The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. The force coefficient of friction* $b = 0N$*-sec/m. Find the equation of motion for the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium point of the spring. Hint: Example 1 of §4.9 NSS gives a worked solution to a similar problem.*

Question (7). *(NSS 4.9.12) A 1/4-kg mass is attached to a spring with stiffness* $k = 8N/m$. *The coefficient of friction for the system is* $b = 2N$*-sec/m. If the mass is pulled 50 cm to the left of equilibrium and given an initial leftward velocity of 2m/sec, when will the mass attain it's maximum displacement to the left? Hint: Example 3 of §4.9 NSS gives a worked solution to a similar problem.*